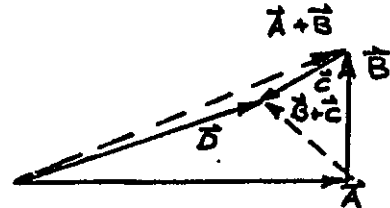
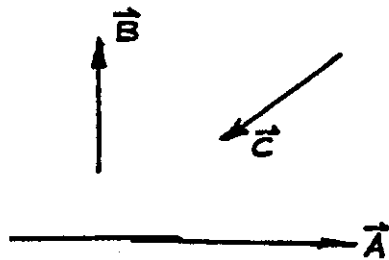


CHAPTER - 2

VECTOR ANALYSIS

Exercise 2.1



$$\vec{B} = (\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

Exercise 2.2

$\vec{A} \cdot \vec{B} = AB \cos \theta$ if $\vec{A} \neq 0$ and $\vec{B} \neq 0$, then
for $\vec{A} \cdot \vec{B} = 0 \Rightarrow \cos \theta = 0$ or $\theta = \pm \frac{\pi}{2}$.

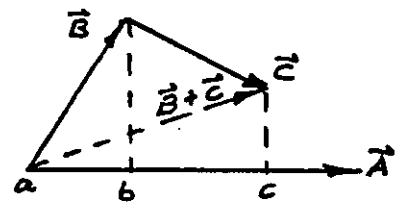
Exercise 2.3

Since $\vec{A} \cdot \vec{B} = AB \cos \theta$, where $B \cos \theta$ is the projection of B onto A .

Thus, $\vec{A} \cdot (\vec{B} + \vec{C})$ is the product of \vec{A} and projection of $\vec{B} + \vec{C}$ on A , which is ac .

However, projection of \vec{B} is ab and that of \vec{C} is bc . Thus,

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$



Exercise 2.4 $|\vec{A} + \vec{B}|^2 = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = \vec{A} \cdot \vec{A} + \vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B}$
 $= A^2 + B^2 + 2AB \cos \theta$

If $\vec{A} \perp \vec{B}$, then $\cos \theta = 0$.

Hence,

$$|\vec{A} + \vec{B}|^2 = A^2 + B^2 \quad \text{if } \vec{A} \perp \vec{B}$$

Exercise 2.5

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

$$\vec{B} + \vec{C} = \vec{D}$$

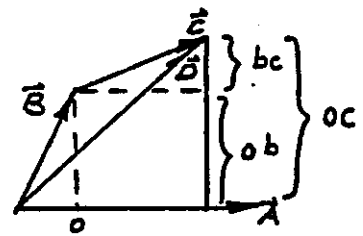
$$\vec{A} \times \vec{B} = A \cdot ob \cdot \vec{a}_n$$

$$\vec{A} \times \vec{C} = A \cdot bc \cdot \vec{a}_n$$

$$\vec{A} \times \vec{B} + \vec{A} \times \vec{C} = A[ob + bc] \vec{a}_n$$

$$= A \cdot oc \cdot \vec{a}_n$$

$$= \vec{A} \times \vec{D} = \vec{A} \times (\vec{B} + \vec{C})$$



\vec{a}_n is unit normal perpendicular to the plane of paper containing \vec{A} , \vec{B} , and \vec{C} .

Exercise 2.6

$$\text{Since } |\vec{A} \times \vec{B}| = AB \sin \theta$$

If $\vec{A} \neq 0$, $\vec{B} \neq 0$ but $|\vec{A} \times \vec{B}| = 0$, then $\sin \theta = 0 \Rightarrow \theta = 0 \Rightarrow \vec{A} \parallel \vec{B}$

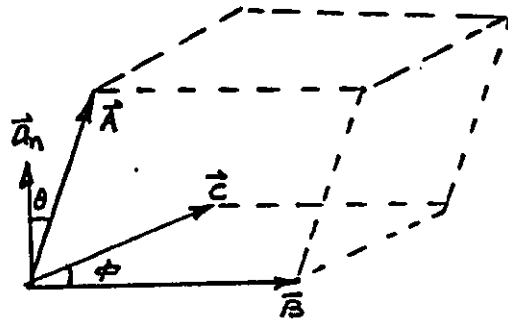
Exercise 2.7

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = ABC \sin \phi \cos \theta$$

is a scalar tripple product.

The scalar tripple product can be permuted cyclically

without changing its value. This can be visualized by constructing a parallelopiped as shown.



Exercise 2.8

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

$$\text{Let } \vec{C} \times \vec{D} = \vec{E}, \text{ then } (\vec{A} \times \vec{B}) \cdot \vec{E} = \vec{A} \cdot (\vec{B} \times \vec{E}) = \vec{A} \cdot [\vec{B} \times (\vec{C} \times \vec{D})]$$

$$= \vec{A} \cdot [(\vec{B} \cdot \vec{D})\vec{C} - (\vec{B} \cdot \vec{C})\vec{D}]$$

$$= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

Exercise 2.9

$$\vec{A} = 2\vec{a}_x + 0.3\vec{a}_y - 1.5\vec{a}_z \quad \text{and} \quad \vec{B} = 10\vec{a}_x + 1.5\vec{a}_y - 7.5\vec{a}_z$$

Since $\vec{B} = 5\vec{A}$, \vec{A} and \vec{B} are dependent vectors.

Exercise 2.10 Position vectors for $P(0, -2, 1)$ and $Q(-2, 0, 3)$ are

$$\vec{r}_1 = \vec{OP} = -2\vec{a}_y + \vec{a}_z \quad \text{and} \quad \vec{r}_2 = \vec{OQ} = -2\vec{a}_x + 3\vec{a}_z$$

Hence, distance vector from P to Q is

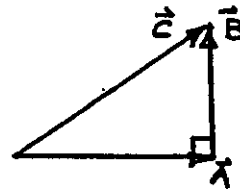
$$\vec{R} = \vec{r}_2 - \vec{r}_1 = -2\vec{a}_x + 2\vec{a}_y + 2\vec{a}_z$$

Exercise 2.11

Note that $\vec{A} + \vec{B} = \vec{C}$, and

$$\vec{A} \cdot \vec{B} = 0. \text{ Thus } \vec{A} \perp \vec{B} \text{ and}$$

\vec{A} , \vec{B} , and \vec{C} form a right angle triangle.



Exercise 2.12

$$\vec{S} + \vec{a} = 3\vec{a}_x + 4\vec{a}_y + 12\vec{a}_z$$

$$|\vec{S} + \vec{a}| = \sqrt{3^2 + 4^2 + 12^2} = 13$$

$$\vec{a}_{|\vec{S}+\vec{a}|} = \frac{3}{13}\vec{a}_x + \frac{4}{13}\vec{a}_y + \frac{12}{13}\vec{a}_z$$

$$\vec{a}_{|\vec{S}+\vec{a}|} \cdot \vec{a}_x = \cos \theta = \frac{3}{13} \Rightarrow \theta = 76.66^\circ$$

Exercise 2.13

$$A_x = A_p \cos \phi - A_\phi \sin \phi, \quad A_y = A_p \sin \phi + A_\phi \cos \phi$$

Thus,

$$A_p = \frac{\begin{vmatrix} A_x & -\sin \phi \\ A_y & \cos \phi \end{vmatrix}}{\begin{vmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{vmatrix}}$$

$$= A_x \cos \phi + A_y \sin \phi$$

$$A_\phi = \frac{\begin{vmatrix} \cos \phi & A_x \\ \sin \phi & A_y \end{vmatrix}}{\begin{vmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{vmatrix}}$$

$$= -A_x \sin \phi + A_y \cos \phi$$

Exercise 2.14

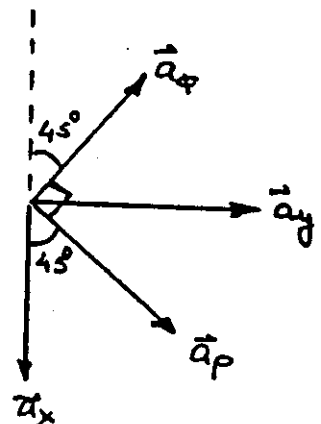
$$\vec{C} = -2\vec{a}_x + 3\vec{a}_y + 4\vec{a}_z$$

$$\begin{aligned} C_p &= \vec{C} \cdot \vec{a}_p = -2\vec{a}_x \cdot \vec{a}_p + 3\vec{a}_y \cdot \vec{a}_p + 4\vec{a}_z \cdot \vec{a}_p \\ &= -2 \cos 45^\circ + 3 \cos 45^\circ = 0.707 \end{aligned}$$

$$\begin{aligned} C_\phi &= \vec{C} \cdot \vec{a}_\phi = -2\vec{a}_x \cdot \vec{a}_\phi + 3\vec{a}_y \cdot \vec{a}_\phi + 4\vec{a}_z \cdot \vec{a}_\phi \\ &= 2 \cos 45^\circ + 3 \cos 45^\circ = 3.535 \end{aligned}$$

$$C_z = \vec{C} \cdot \vec{a}_z = 4$$

$$\vec{C} = 0.707 \vec{a}_p + 3.535 \vec{a}_\phi + 4 \vec{a}_z$$



Exercise 2.15

$$x = r \cos \phi \quad y = r \sin \phi \quad r = \sqrt{x^2 + y^2}$$

$$a. \vec{F} = r \sin \phi \vec{a}_\rho - r \cos \phi \vec{a}_\phi$$

$$b. \vec{H} = \frac{1}{r} \vec{a}_\phi \vec{a}_\rho$$

Using (2.38),

$$A_x = 2r \cos \phi \sin \phi = \frac{2xy}{\sqrt{x^2 + y^2}}$$

$$H_x = \frac{1}{r} \cos \phi = \frac{x}{x^2 + y^2}$$

$$A_y = r(\sin^2 \phi - \cos^2 \phi) = \frac{y^2 - x^2}{\sqrt{x^2 + y^2}}$$

$$H_y = \frac{1}{r} \sin \phi = \frac{y}{x^2 + y^2}$$

$$A_z = 0$$

$$H_z = 0$$

Exercise 2.16 Transform: $P(1, \pi, 0) \Rightarrow P(-1, 0, 0)$

$$Q(0, -\frac{\pi}{2}, 2) \Rightarrow Q(0, 0, 2)$$

$$\vec{PQ} = \vec{R} = \vec{a}_x + 2\vec{a}_z \Rightarrow R = \sqrt{5}$$

$$\vec{QP} = -\vec{PQ} = -\vec{a}_x - 2\vec{a}_z$$

Exercise 2.17

$$x = R \sin \theta \cos \phi, \quad y = R \sin \theta \sin \phi, \quad z = R \cos \theta$$

$$A_r = x \sin \theta \cos \phi + y \sin \theta \sin \phi + z \cos \theta$$

$$= R \sin^2 \theta \cos^2 \phi + R \sin^2 \theta \sin^2 \phi + R \cos^2 \theta = R$$

$$A_\theta = x \cos \theta \cos \phi + y \cos \theta \sin \phi - z \sin \theta$$

$$= R \sin \theta \cos \theta \cos^2 \phi + R \sin \theta \cos \theta \sin^2 \phi - R \sin \theta \cos \theta = 0$$

$$A_\phi = -x \sin \phi + y \cos \phi$$

$$= -R \sin \theta \sin \phi \cos \phi + R \sin \theta \sin \phi \cos \phi = 0$$

$$\text{Hence } \vec{F} = R \vec{a}_r$$

Exercise 2.18 $\vec{F} = r \vec{a}_r + r \tan \theta \vec{a}_\theta + r \sin \theta \cos \phi \vec{a}_\phi$

$$F_x = r \sin \theta \cos \phi + r \frac{\sin \theta}{\cos \theta} \cos \theta \cos \phi - r \sin \theta \sin \phi \cos \phi$$

$$= 2r \sin \theta \cos \phi - r \sin \theta \cos \phi \sin \phi = 2x - \frac{xy}{\sqrt{x^2 + y^2}}$$

$$F_y = r \sin \theta \sin \phi + r \frac{\sin \theta}{\cos \theta} \cos \theta \sin \phi + r \sin \theta \cos^2 \phi$$

$$= 2y + \frac{x^2}{\sqrt{x^2 + y^2}}$$

$$F_z = r \cos \theta - r \frac{\sin^2 \theta}{\cos \theta} = z - \frac{x^2 + y^2}{z} = \frac{z^2 - x^2 - y^2}{z} = 2z - \frac{x^2 + y^2 + z^2}{z}$$

$$\vec{F} = \left[2x - \frac{xy}{\sqrt{x^2 + y^2}} \right] \vec{a}_x + \left[2y + \frac{x^2}{\sqrt{x^2 + y^2}} \right] \vec{a}_y + \left[2z - \frac{x^2 + y^2 + z^2}{z} \right] \vec{a}_z$$

Exercise 2.19

$$P(2, \pi/2, 3\pi/4) \Rightarrow P(-1.414, 1.414, 0)$$

$$Q(10, \pi/4, \pi/2) \Rightarrow Q(0, 7.07, 7.07)$$

$$\vec{R} = \vec{PQ} = 1.414 \vec{a}_x + 5.656 \vec{a}_y + 7.07 \vec{a}_z \quad |\vec{PQ}| = 9.164$$

Exercise 2.20

$$\vec{S} = 12 \vec{a}_r + 5 \vec{a}_\theta + \pi \vec{a}_\phi \quad @ (2, \pi, \pi/2) \Rightarrow \vec{S} = -\pi \vec{a}_x - 5 \vec{a}_y - 12 \vec{a}_z$$

$$\vec{T} = 2 \vec{a}_r + 0.5\pi \vec{a}_\theta \quad @ (5, \pi/2, \pi/2) \Rightarrow \vec{T} = 2 \vec{a}_y - 0.5\pi \vec{a}_z$$

$$\vec{S} + \vec{T} = -\pi \vec{a}_x - 3 \vec{a}_y - 13.57 \vec{a}_z, \quad \vec{S} \cdot \vec{T} = 0 - 10 + 6\pi = 8.85$$

$$\vec{S} \times \vec{T} = 31.85 \vec{a}_x - 4.93 \vec{a}_y - 6.28 \vec{a}_z \Rightarrow \vec{a}_{\vec{S} \times \vec{T}} = \frac{\vec{S} \times \vec{T}}{|\vec{S} \times \vec{T}|}$$

$$= 0.47 \vec{a}_x - 0.15 \vec{a}_y - 0.19 \vec{a}_z$$

Exercise 2.21

$$\text{Since } g = g(u(t), v(t), s(t)),$$

$$\frac{dg}{dt} = \frac{\partial g}{\partial u} \frac{du}{dt} + \frac{\partial g}{\partial v} \frac{dv}{dt} + \frac{\partial g}{\partial s} \frac{ds}{dt}$$

Exercise 2.22

$$\text{For } G(x, y, z, t)$$

$$\frac{dG}{dt} = \frac{\partial G}{\partial x} \frac{dx}{dt} + \frac{\partial G}{\partial y} \frac{dy}{dt} + \frac{\partial G}{\partial z} \frac{dz}{dt} + \frac{\partial G}{\partial t}$$

Exercise 2.23

$$\frac{\partial \vec{F}}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{\vec{F}(x, y + \Delta y, z) - \vec{F}(x, y, z)}{\Delta y}$$

$$\frac{\partial \vec{F}}{\partial z} = \lim_{\Delta z \rightarrow 0} \frac{\vec{F}(x, y, z + \Delta z) - \vec{F}(x, y, z)}{\Delta z}$$

Exercise 2.24

$$\vec{r} = x \vec{a}_x + y \vec{a}_y + z \vec{a}_z \Rightarrow d\vec{r} = dx \vec{a}_x + dy \vec{a}_y + dz \vec{a}_z$$

Exercise 2.25

$$\vec{l} = \rho \vec{a}_\rho + z \vec{a}_z = \rho \cos \phi \vec{a}_x + \rho \sin \phi \vec{a}_y + z \vec{a}_z$$

$$d\vec{l} = d\rho \cos \phi \vec{a}_x - \rho \sin \phi d\phi \vec{a}_x + d\rho \sin \phi \vec{a}_y + \rho \cos \phi d\phi \vec{a}_y + dz \vec{a}_z$$

$$= (\cos \phi \vec{a}_x + \sin \phi \vec{a}_y) d\rho + (\cos \phi \vec{a}_y - \sin \phi \vec{a}_x) \rho d\phi + dz \vec{a}_z$$

$$= d\rho \vec{a}_\rho + \rho d\phi \vec{a}_\phi + dz \vec{a}_z$$

Exercise 2.26

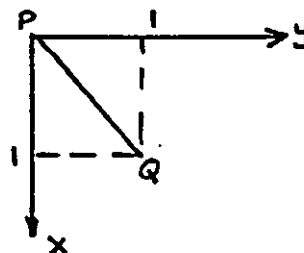
$$\begin{aligned}
 \vec{r} &= r \vec{a}_r = r \sin \theta \cos \phi \vec{a}_x + r \sin \theta \sin \phi \vec{a}_y + r \cos \theta \vec{a}_z \\
 d\vec{r} &= \vec{a}_x [\sin \theta \cos \phi dr + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi] \\
 &\quad + \vec{a}_y [\sin \theta \sin \phi dr + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi] \\
 &\quad + \vec{a}_z [\cos \theta dr - r \sin \theta d\theta] \\
 &= [\sin \theta \cos \phi \vec{a}_x + \sin \theta \sin \phi \vec{a}_y + \cos \theta \vec{a}_z] dr \\
 &\quad + [\cos \theta \cos \phi \vec{a}_x + \cos \theta \sin \phi \vec{a}_y - \sin \theta \vec{a}_z] r d\theta \\
 &\quad + [-\sin \phi \vec{a}_x + \cos \phi \vec{a}_y] r \sin \theta d\phi \\
 &= dr \vec{a}_r + r d\theta \vec{a}_\theta + r \sin \theta d\phi \vec{a}_\phi
 \end{aligned}$$

Exercise 2.27

$$g = 20xy \quad d\vec{r} = dx \vec{a}_x + dy \vec{a}_y + dz \vec{a}_z$$

$$\begin{aligned}
 \text{a) } \int_P^Q g d\vec{r} &= \int_P^Q 20xy (dx \vec{a}_x + dy \vec{a}_y) \\
 &= \int_{x=0}^1 20x^2 (\vec{a}_x + \vec{a}_y) dx = \frac{20}{3} [\vec{a}_x + \vec{a}_y]
 \end{aligned}$$

$$\begin{aligned}
 z &= 0 \\
 dz &= 0 \\
 y &= x \\
 dy &= dx
 \end{aligned}$$



$$\begin{aligned}
 \text{b) } \int_{x=0}^1 20x(4x^2) [dx \vec{a}_x + 8x dx \vec{a}_y] &\quad y = 4x^2 \quad dy = 8x dx \\
 &= 20 \vec{a}_x + 128 \vec{a}_y
 \end{aligned}$$

Exercise 2.28

$$\vec{r} \cdot d\vec{r} = (b \vec{a}_r) \cdot (b d\phi \vec{a}_\phi) = 0 \quad \because \vec{a}_r \cdot \vec{a}_\phi = 0$$

$$\text{Hence } \oint_C \vec{r} \cdot d\vec{r} = 0$$

Exercise 2.29

$$\begin{aligned}
 \vec{r} \cdot d\vec{s} &= (b \vec{a}_r) \cdot (\vec{a}_r b^2 \sin \theta d\theta d\phi) \\
 &= b^3 \sin \theta d\theta d\phi
 \end{aligned}$$

$$\begin{aligned}
 \oint \vec{r} \cdot d\vec{s} &= b^3 \int_{\theta=0}^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \\
 &= 4\pi b^3
 \end{aligned}$$

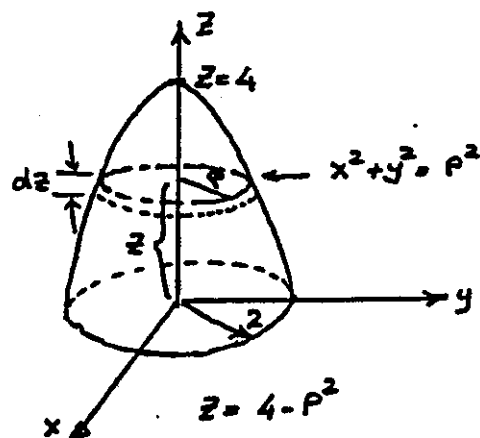
Exercise 2.30

$$z=0 \Rightarrow x^2+y^2=4$$

$$x=0, y=0 \Rightarrow z=4$$

$$\begin{aligned} V &= \int_0^4 \int_0^{2\pi} \int_0^{\sqrt{4-z}} \rho d\rho d\phi dz \\ &= 2\pi \int_0^4 \int_0^{\sqrt{4-z}} \rho d\rho dz \\ &= 8\pi \end{aligned}$$

$$\int_0^{2\pi} d\phi = 2\pi$$



Exercise 2.31

$$df = \nabla f \cdot d\vec{r}$$

$$\left. \begin{aligned} \vec{a}_x &= \cos\phi \vec{a}_\rho - \sin\phi \vec{a}_\phi \\ \vec{a}_y &= \sin\phi \vec{a}_\rho + \cos\phi \vec{a}_\phi \end{aligned} \right\} \textcircled{2}$$

$$\nabla = \vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z} \textcircled{1}$$

$$x = \rho \cos\phi \Rightarrow dx = \cos\phi d\rho - \rho \sin\phi d\phi$$

$$y = \rho \sin\phi \Rightarrow dy = \sin\phi d\rho + \rho \cos\phi d\phi$$

$$\text{Thus, } d\rho = \cos\phi dx + \sin\phi dy \text{ and } d\phi = -\frac{1}{\rho} \sin\phi dx + \frac{1}{\rho} \cos\phi dy \textcircled{3}$$

$$\text{But } d\rho = \frac{\partial \rho}{\partial x} dx + \frac{\partial \rho}{\partial y} dy \text{ and } d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \textcircled{4}$$

From ③ and ④

$$\frac{\partial \rho}{\partial x} = \cos\phi, \quad \frac{\partial \rho}{\partial y} = \sin\phi, \quad \frac{\partial \phi}{\partial x} = -\frac{1}{\rho} \sin\phi, \text{ and } \frac{\partial \phi}{\partial y} = \frac{1}{\rho} \cos\phi \textcircled{5}$$

However,

$$\left. \begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial}{\partial \phi} \frac{\partial \phi}{\partial x} = \cos\phi \frac{\partial}{\partial \rho} - \frac{1}{\rho} \sin\phi \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial \rho} \frac{\partial \rho}{\partial y} + \frac{\partial}{\partial \phi} \frac{\partial \phi}{\partial y} = \sin\phi \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cos\phi \frac{\partial}{\partial \phi} \end{aligned} \right\} \textcircled{6}$$

Substitute ⑤ and ⑥ in ①

$$\begin{aligned} \nabla &= (\cos\phi \vec{a}_\rho - \sin\phi \vec{a}_\phi) \left(\cos\phi \frac{\partial}{\partial \rho} - \frac{1}{\rho} \sin\phi \frac{\partial}{\partial \phi} \right) \\ &\quad + (\sin\phi \vec{a}_\rho + \cos\phi \vec{a}_\phi) \left(\sin\phi \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cos\phi \frac{\partial}{\partial \phi} \right) + \vec{a}_z \frac{\partial}{\partial z} \\ &= \vec{a}_\rho \frac{\partial}{\partial \rho} + \vec{a}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \vec{a}_z \frac{\partial}{\partial z} \end{aligned}$$

Exercise 2.32

$$\vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z = \rho\vec{a}_\rho + z\vec{a}_z = r\vec{a}_r$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2} = r$$

$$\nabla r = \frac{\partial r}{\partial x}\vec{a}_x + \frac{\partial r}{\partial y}\vec{a}_y + \frac{\partial r}{\partial z}\vec{a}_z \quad \text{However, } \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$= \frac{x}{r}\vec{a}_x + \frac{y}{r}\vec{a}_y + \frac{z}{r}\vec{a}_z = \frac{\vec{r}}{r} = \vec{a}_r$$

$$\nabla r = \frac{\partial r}{\partial r}\vec{a}_r = \vec{a}_r$$

Exercise 2.33

$$f = 12x^2 + yz^2 \Rightarrow \nabla f = 24x\vec{a}_x + z^2\vec{a}_y + 2yz\vec{a}_z$$

$$\text{at } P(-1, 0, 1), \nabla f = -24\vec{a}_x + \vec{a}_y \quad Q = (1, 1, 1)$$

$$(\nabla f)_x = \nabla f \cdot \vec{a}_x = 24x = -24 \text{ at } P(-1, 0, 1)$$

$$\vec{F}_Q = 2\vec{a}_x + \vec{a}_y$$

$$(\nabla f)_y = \nabla f \cdot \vec{a}_y = z^2 = 1 \text{ at } P(-1, 0, 1)$$

$$\vec{a}_{PQ} = \frac{2}{\sqrt{5}}\vec{a}_x + \frac{1}{\sqrt{5}}\vec{a}_y$$

$$(\nabla f)_z = \nabla f \cdot \vec{a}_z = 2yz = 0 \text{ at } P(-1, 0, 1)$$

$$(\nabla f)_{PQ} = \nabla f \cdot \vec{a}_{PQ} = \frac{48}{\sqrt{5}}x + \frac{1}{\sqrt{5}}z^2 = -21.02 \text{ at } P(-1, 0, 1)$$

Exercise 2.34

$$v = \rho\vec{a}_\rho + z\vec{a}_z = r\vec{a}_r$$

$$\nabla \cdot \vec{v} = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho^2) + \frac{\partial}{\partial z}z = 3$$

$$\nabla \cdot \vec{r} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^3) = 3$$

Exercise 2.35

$$\vec{F} = -xy\vec{a}_x + 3x^2yz\vec{a}_y + xz^3\vec{a}_z$$

$$\nabla \cdot \vec{F} = -y + 3x^2z + 3z^3x = 19 \text{ at } P(1, -1, 2)$$

Exercise 2.36

$$\nabla \cdot \vec{r} = 3$$

$$\vec{F} = r\vec{a}_r$$

$$\nabla \cdot (r^2\vec{a}_r) = \frac{1}{r^2} \frac{\partial}{\partial r}(r^4) = 4r = (2+2)r^{2-1}$$

$$\nabla \cdot (r^3\vec{a}_r) = \frac{1}{r^2} \frac{\partial}{\partial r}(r^5) = 5r^2 = (3+2)r^{3-1}$$

$$\nabla \cdot (r^4\vec{a}_r) = \frac{1}{r^2} \frac{\partial}{\partial r}(r^6) = 6r^3 = (4+2)r^{4-1}$$

⋮

$$\nabla \cdot (r^n\vec{a}_r) = \frac{1}{r^2} \frac{\partial}{\partial r}(r^{n+2}) = (n+2)r^{n-1}$$

Exercise 2.37 $\vec{F} = x \vec{a}_x + xy \vec{a}_y + xyz \vec{a}_z \Rightarrow \nabla \cdot \vec{F} = 1 + x + xy$

$$\begin{aligned} \int_V \nabla \cdot \vec{F} dV &= \int_V (1 + x + xy) dV = \int_V (1 + r \sin \theta \cos \phi + r^2 \sin^2 \theta \sin \phi \cos \phi) dV \\ &= \int_0^2 r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi + \int_0^2 r^3 dr \int_0^\pi \sin^2 \theta d\theta \int_0^{2\pi} \cos \phi d\phi \\ &\quad + \int_0^2 r^4 dr \int_0^\pi \sin^3 \theta d\theta \int_0^{2\pi} \sin \phi \cos \phi d\phi = \frac{32}{3} \pi \end{aligned}$$

$\vec{F} \cdot d\vec{s} = F_r r^2 \sin \theta d\theta d\phi$ at $r=2$

$x = 2 \sin \theta \cos \phi$

$F_r = x \sin \theta \cos \phi + xy \sin \theta \sin \phi + xyz \cos \theta$

$y = 2 \sin \theta \sin \phi$

$= 2 \sin^2 \theta \cos^2 \phi + 4 \sin^3 \theta \sin^2 \phi \cos \phi$

$z = 2 \cos \theta$

$$\begin{aligned} \oint_S \vec{F} \cdot d\vec{s} &= 8 \int_0^\pi \sin^3 \theta d\theta \int_0^{2\pi} \cos^2 \phi d\phi + 16 \int_0^\pi \sin^4 \theta d\theta \int_0^{2\pi} \sin^2 \phi \cos \phi d\phi \\ &\quad + 32 \int_0^\pi \sin^3 \theta \cos^2 \theta d\theta \int_0^{2\pi} \sin \phi \cos \phi d\phi \\ &= 8 \int_0^\pi (1 - \cos^2 \theta) \sin \theta d\theta \int_0^{2\pi} \frac{1}{2} (1 + \cos 2\phi) d\phi = \frac{32}{3} \pi \end{aligned}$$

Exercise 2.38

$\vec{F} = F_x \vec{a}_x + F_y \vec{a}_y + F_z \vec{a}_z$

$\int_{C_1} \vec{F} \cdot d\vec{l} = F_y \Delta y \big|_{at z}$

$\int_{C_2} \vec{F} \cdot d\vec{l} = F_z \Delta z \big|_{at y+\Delta y}$

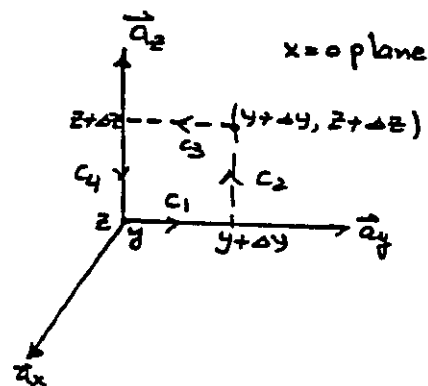
$\int_{C_3} \vec{F} \cdot d\vec{l} = -F_y \Delta y \big|_{at z+\Delta z}$

$\int_{C_4} \vec{F} \cdot d\vec{l} = -F_z \Delta z \big|_{at y}$

$F_z \Delta z \big|_{y+\Delta y} - F_z \Delta z \big|_y = \frac{\partial F_z}{\partial y} \Delta y \Delta z$

$F_y \Delta y \big|_{z+\Delta z} - F_y \Delta y \big|_z = \frac{\partial F_y}{\partial z} \Delta y \Delta z$

$\oint_C \vec{F} \cdot d\vec{l} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \Delta y \Delta z$



Thus, $(\nabla \times \vec{F}) \cdot \vec{a}_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$

Similarly, we can prove that

$(\nabla \times \vec{F}) \cdot \vec{a}_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}$

and

$(\nabla \times \vec{F}) \cdot \vec{a}_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$

Exercise 2.39 Cylindrical coordinate system

$$\int_{C_1} \vec{F} \cdot d\vec{l} = F_\phi \rho d\phi \Big|_z \quad d\vec{s} = \rho d\phi dz \vec{a}_\rho$$

$$\int_{C_3} \vec{F} \cdot d\vec{l} = -F_\phi \rho d\phi \Big|_{z+dz} = - \left[F_\phi + \frac{\partial F_\phi}{\partial z} dz \right] \rho d\phi$$

$$\int_{C_1} \vec{F} \cdot d\vec{l} + \int_{C_3} \vec{F} \cdot d\vec{l} = - \frac{\partial F_\phi}{\partial z} \rho d\phi dz$$

$$\int_{C_2} \vec{F} \cdot d\vec{l} = F_z dz \Big|_{\phi+d\phi} = (F_z + \frac{\partial F_z}{\partial \phi} d\phi) dz$$

$$\int_{C_2} \vec{F} \cdot d\vec{l} + \int_{C_4} \vec{F} \cdot d\vec{l} = \frac{\partial F_z}{\partial \phi} d\phi dz \Rightarrow \oint_C \vec{F} \cdot d\vec{l} = \frac{\partial F_z}{\partial \phi} d\phi dz - \frac{\partial F_\phi}{\partial z} \rho d\phi dz$$

$$\text{Thus, } (\nabla \times \vec{F}) \cdot \vec{a}_\phi = \lim_{ds \rightarrow 0} \frac{\oint_C \vec{F} \cdot d\vec{l}}{ds} = \frac{1}{\rho} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}$$

\vec{a}_ϕ -component:

$$\int_{C_1} \vec{F} \cdot d\vec{l} = F_z dz \Big|_\rho \quad \int_{C_4} \vec{F} \cdot d\vec{l} = -F_\rho d\rho \Big|_z$$

$$\int_{C_3} \vec{F} \cdot d\vec{l} = -F_z dz \Big|_{\rho+d\rho} = - \left[F_z + \frac{\partial F_z}{\partial \rho} d\rho \right] dz$$

$$\int_{C_2} \vec{F} \cdot d\vec{l} = F_\rho d\rho \Big|_{z+dz} = \left[F_\rho + \frac{\partial F_\rho}{\partial z} dz \right] d\rho$$

$$\oint_C \vec{F} \cdot d\vec{l} = \frac{\partial F_\rho}{\partial z} dz d\rho - \frac{\partial F_z}{\partial \rho} d\rho dz$$

$$\text{Thus } (\nabla \times \vec{F}) \cdot \vec{a}_\phi = \frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}$$

\vec{a}_z -component:

$$\int_{C_1} \vec{F} \cdot d\vec{l} = F_\phi d\rho \Big|_\phi$$

$$\int_{C_3} \vec{F} \cdot d\vec{l} = -F_\phi d\rho \Big|_{\phi+d\phi} = - \left[F_\phi + \frac{\partial F_\phi}{\partial \phi} d\phi \right] d\rho$$

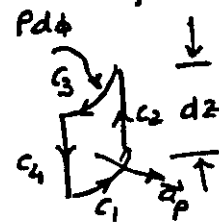
$$\int_{C_2} \vec{F} \cdot d\vec{l} = F_\phi (\rho+d\rho) d\phi = \left(F_\phi + \frac{\partial F_\phi}{\partial \rho} d\rho \right) (\rho+d\rho) d\phi$$

$$\int_{C_4} \vec{F} \cdot d\vec{l} = -F_\phi \rho d\phi$$

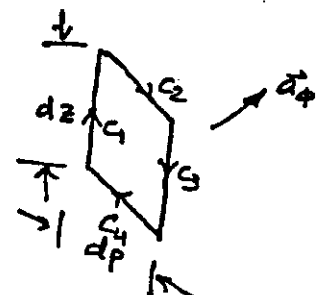
$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{l} &= F_\phi d\rho d\phi + \rho \frac{\partial F_\phi}{\partial \rho} d\rho d\phi - \frac{\partial F_\phi}{\partial \phi} \rho d\rho d\phi \\ &= \frac{\partial}{\partial \rho} (\rho F_\phi) d\rho d\phi - \frac{\partial F_\phi}{\partial \phi} \rho d\rho d\phi \end{aligned}$$

$$\text{Thus } (\nabla \times \vec{F}) \cdot \vec{a}_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho F_\phi) - \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi}$$

\vec{a}_ρ -component

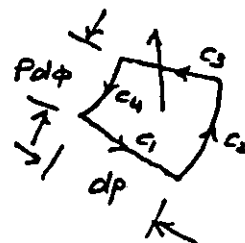


$$\int_{C_4} \vec{F} \cdot d\vec{l} = -F_z dz \Big|_\phi$$



$$d\vec{s} = dz d\rho \vec{a}_\phi$$

$$d\vec{s} = \rho d\rho d\phi \vec{a}_z$$



Exercise 2.40 $\vec{F} = (x/r) \vec{a}_x$ $r = \sqrt{x^2 + y^2 + z^2}$

$$\nabla \times \vec{F} = \vec{a}_y \frac{\partial F_x}{\partial z} - \vec{a}_z \frac{\partial F_x}{\partial y} = \frac{x}{r^3} [-z \vec{a}_y + y \vec{a}_z]$$

Exercise 2.41 Let $\vec{F} = F_x \vec{a}_x + F_y \vec{a}_y + F_z \vec{a}_z$, then

$$\begin{aligned} \nabla \cdot (\nabla \times \vec{F}) &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \frac{\partial}{\partial x} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \\ &\quad + \frac{\partial}{\partial z} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\ &= \frac{\partial^2 F_z}{\partial x \partial y} - \frac{\partial^2 F_y}{\partial x \partial z} + \frac{\partial^2 F_x}{\partial y \partial z} - \frac{\partial^2 F_z}{\partial y \partial x} + \frac{\partial^2 F_y}{\partial z \partial x} - \frac{\partial^2 F_x}{\partial z \partial y} = 0 \end{aligned}$$

Exercise 2.42 Let us use cylindrical coordinate system

$$\nabla f = \frac{\partial f}{\partial \rho} \vec{a}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \vec{a}_\phi + \frac{\partial f}{\partial z} \vec{a}_z$$

$$\begin{aligned} \nabla \times \nabla f &= \frac{1}{\rho} \begin{vmatrix} \vec{a}_\rho & \rho \vec{a}_\phi & \vec{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \phi} & \frac{\partial f}{\partial z} \end{vmatrix} = \frac{1}{\rho} \vec{a}_\rho \left(\frac{\partial^2 f}{\partial \phi \partial z} - \frac{\partial^2 f}{\partial z \partial \phi} \right) \\ &\quad + \vec{a}_\phi \left(\frac{\partial^2 f}{\partial z \partial \rho} - \frac{\partial^2 f}{\partial \rho \partial z} \right) \\ &\quad + \frac{1}{\rho} \vec{a}_z \left(\frac{\partial^2 f}{\partial \rho \partial \phi} - \frac{\partial^2 f}{\partial \phi \partial \rho} \right) = 0 \end{aligned}$$

Exercise 2.43

$$\vec{F} = 10 \cos \theta \vec{a}_r - 10 \sin \theta \vec{a}_\theta \Rightarrow \nabla \times \vec{F} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \vec{a}_r & r \vec{a}_\theta & r \sin \theta \vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 10 \cos \theta & -10 r \sin \theta & 0 \end{vmatrix}$$

$$\text{Thus } \int_S \nabla \times \vec{F} \cdot d\vec{S} = 0$$

$$= 0$$

On the circular path c , $d\vec{l} = a d\phi \vec{a}_\phi$ at $r=a$

$$\vec{F} \cdot d\vec{l} = (10 \cos \theta \vec{a}_r - 10 \sin \theta \vec{a}_\theta) \cdot a d\phi \vec{a}_\phi = 0$$

$$\text{Hence } \oint_c \vec{F} \cdot d\vec{l} = 0$$

Exercise 2.44

$$g = 25x^2yz + 12xy^2$$

$$\nabla g = (50xyz + 12y^2)\vec{a}_x + (25x^2z + 24xy)\vec{a}_y + 25x^2y\vec{a}_z$$

$$\nabla \cdot \nabla g = 50yz + 24x$$

$$\begin{aligned}\nabla^2 g &= \frac{\partial^2}{\partial x^2}(25x^2yz + 12y^2) + \frac{\partial^2}{\partial y^2}(25x^2yz + 12xy^2) + \frac{\partial^2}{\partial z^2}(25x^2yz + 12xy^2) \\ &= 50yz + 24x\end{aligned}$$

Exercise 2.45 $f = 2x^2y^3 + 3yz^3 \Rightarrow \nabla^2 f = 4y^3 + 12x^2y + 18yz$

$$\nabla f = 4xy^3\vec{a}_x + (6x^2y^2 + 3z^3)\vec{a}_y + 9yz^2\vec{a}_z$$

$$\nabla \cdot \nabla f = 4y^3 + 12x^2y + 18yz$$

Exercise 2.46 $h = \rho^2 \sin 2\phi + z^3 \cos \phi$

$$\nabla h = 2\rho \sin 2\phi \vec{a}_\rho + (2\rho \cos 2\phi - \frac{z^3}{\rho} \sin \phi) \vec{a}_\phi + 3z^2 \cos \phi \vec{a}_z$$

$$\nabla \cdot \nabla h = 4 \sin 2\phi - 4 \sin 2\phi - \frac{z^3}{\rho^2} \cos \phi + 6z \cos \phi = 6z \cos \phi - \frac{z^3}{\rho^2} \cos \phi$$

$$\begin{aligned}\nabla^2 h &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho \frac{\partial}{\partial \rho} (\rho^2 \sin 2\phi + z^3 \cos \phi) \right] + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} (\rho^2 \sin 2\phi + z^3 \cos \phi) + \frac{\partial^2}{\partial z^2} (z^3 \cos \phi) \\ &= 4 \sin 2\phi - 4 \sin 2\phi - \frac{z^3}{\rho^2} \cos \phi + 6z \cos \phi = 6z \cos \phi - \frac{z^3}{\rho^2} \cos \phi\end{aligned}$$

Exercise 2.47 $\Phi = k[\ln b - \ln \rho]$

$$\begin{aligned}\nabla^2 \Phi &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho \frac{\partial}{\partial \rho} (k \ln b - k \ln \rho) \right] + \cancel{\frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2}} + \cancel{\frac{\partial^2 \Phi}{\partial z^2}} \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho \left(-\frac{k}{\rho} \right) \right] = 0\end{aligned}$$

Exercise 2.48 $\nabla \Phi = \frac{\partial \Phi}{\partial \rho} \vec{a}_\rho = -\frac{k}{\rho} \vec{a}_\rho \quad \nabla^2 \Phi = 0$

$$\int_V \cancel{\Phi \nabla^2 \Phi} d\tau + \int_V |\nabla \Phi|^2 d\tau = \oint_S \Phi \nabla \Phi \cdot \vec{ds}$$

$$\int_V |\nabla \Phi|^2 d\tau = k^2 \int_a^b \frac{1}{\rho^2} \rho d\rho \int_0^{2\pi} d\phi \int_0^1 dz = 2\pi k^2 \ln\left(\frac{b}{a}\right) \quad (1)$$

at surface $\rho = b$ $\Phi = k \ln(b/b) = 0 \Rightarrow$ no contribution from this surface

at surface $\rho = a$ $\Phi = k \ln(b/a)$ and $\nabla \Phi = -\frac{k}{a} \vec{a}_\rho$, $\vec{ds} = -a d\phi dz \vec{a}_\rho$

$$\oint_S \Phi \nabla \Phi \cdot \vec{ds} = k \ln\left(\frac{b}{a}\right) \int_0^{2\pi} \int_0^1 \frac{k}{a} a d\phi dz = 2\pi k^2 \ln(b/a) \quad (2)$$

note: (1) = (2) Green's Theorem satisfied.

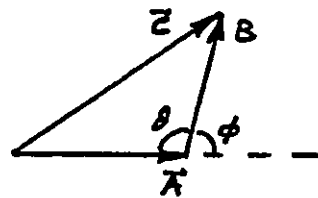
Problem 2.1

$$\vec{c} = \vec{A} + \vec{B}$$

$$c^2 = \vec{c} \cdot \vec{c} = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B})$$

$$= A^2 + B^2 + 2AB \cos \phi$$

$$= A^2 + B^2 - 2AB \cos \theta$$



Problem 2.2 $\vec{D} = \vec{B} \times \vec{c}$ is a vector normal to \vec{B} and \vec{c} . Thus, $\vec{D} \perp \vec{A}$. Hence $\vec{D} \cdot \vec{A} = 0$

Problem 2.3 $\vec{R} = (x-2)\vec{a}_x + (y-3)\vec{a}_y + (z-4)\vec{a}_z$

$$\vec{R} \cdot \vec{R} = R^2 = (x-2)^2 + (y-3)^2 + (z-4)^2$$

R is the radius of the sphere.

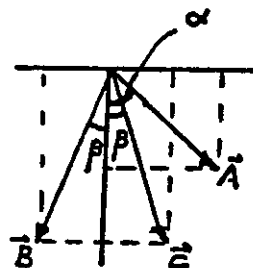
Problem 2.4 $\vec{A} = \cos \alpha \vec{a}_x + \sin \alpha \vec{a}_y$

$$A = \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{\cos^2 \alpha + \sin^2 \alpha} = 1$$

$$\beta < \alpha$$

$$\vec{B} = \cos \beta \vec{a}_x - \sin \beta \vec{a}_y \Rightarrow B = 1$$

$$\vec{c} = \cos \beta \vec{a}_x + \sin \beta \vec{a}_y \Rightarrow c = 1$$



$$\vec{A} \times \vec{B} = -\vec{a}_z AB \sin(\alpha + \beta)$$

$$\vec{A} \times \vec{c} = -\vec{a}_z AC \sin(\alpha - \beta)$$

$$= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \cos \alpha & \sin \alpha & 0 \\ \cos \beta & -\sin \beta & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \cos \alpha & \sin \alpha & 0 \\ \cos \beta & \sin \beta & 0 \end{vmatrix}$$

$$= -\vec{a}_z [\sin \alpha \cos \beta + \cos \alpha \sin \beta]$$

$$= -\vec{a}_z [\sin \alpha \cos \beta - \cos \alpha \sin \beta]$$

Hence, $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

Problem 2.5

$$\vec{A} = \vec{a}_x + \vec{a}_y + \vec{a}_z, \quad \vec{B} = 4\vec{a}_x + 4\vec{a}_y + \vec{a}_z$$

$$\vec{D} = \vec{B} - \vec{A} = 3\vec{a}_x + 3\vec{a}_y$$

$$D = \sqrt{3^2 + 3^2} = 4.24$$

Problem 2.6 $\vec{A} = 3\vec{a}_x + 2\vec{a}_y - \vec{a}_z$, $\vec{B} = \vec{a}_x - 2\vec{a}_y + 3\vec{a}_z$

$$\vec{A} + \vec{B} = 4\vec{a}_x + 2\vec{a}_z \quad \vec{A} \cdot \vec{B} = 3 - 4 - 3 = -4$$

$$\vec{C} = \vec{A} \times \vec{B} = 4\vec{a}_x - 10\vec{a}_y - 8\vec{a}_z = C\vec{a}_n \quad C = \sqrt{4^2 + 10^2 + 8^2} = 13.416$$

$$\vec{a}_n = 0.298\vec{a}_x - 0.745\vec{a}_y - 0.596\vec{a}_z, \quad A = 3.742, \quad B = 3.742$$

$$\cos \alpha = \frac{\vec{A} \cdot \vec{B}}{AB} = -0.286 \Rightarrow \alpha = 106.6^\circ \text{ or } \alpha = -106.6^\circ$$

$$\sin \alpha = \frac{|\vec{A} \times \vec{B}|}{AB} = \frac{C}{AB} = 0.958 \Rightarrow \alpha = 73.4^\circ \text{ or } \alpha = 106.6^\circ \} \Rightarrow \alpha = 106.6^\circ$$

Scalar projection of \vec{A} onto B : $\vec{A} \cdot \vec{a}_B = \frac{\vec{A} \cdot \vec{B}}{B} = -1.069$

Vector projection of A onto B : $-1.069\vec{a}_B = -1.069\frac{\vec{B}}{B}$
 $= -0.286[\vec{a}_x - 2\vec{a}_y + 3\vec{a}_z]$

Problem 2.7 $\vec{r}_1 = \vec{OP} = 5\vec{a}_x + 12\vec{a}_y + \vec{a}_z$, $\vec{r}_2 = \vec{OQ} = 2\vec{a}_x - 3\vec{a}_y + \vec{a}_z$

$$\vec{PQ} = \vec{r} = \vec{r}_2 - \vec{r}_1 = -3\vec{a}_x - 15\vec{a}_y \Rightarrow r = PQ = 15.3$$

Since \vec{PQ} has no z -component, it is in the xy plane.

$P(5, 12, 1)$ and $Q(2, -3, 1)$.

Problem 2.8 Since $\vec{A} + \vec{C} = \vec{B}$ they form a triangle.

Since $\vec{A} \cdot \vec{C} = 0$, $\vec{A} \perp \vec{C}$.

$$\text{Area} = \frac{1}{2} |\vec{A} \times \vec{C}| = \frac{1}{2} |5\vec{a}_x - 5\vec{a}_y - 20\vec{a}_z| = 10.6$$

Problem 2.9 $\vec{A} \cdot \vec{B} = 30 + 10 - 40 = 0 \Rightarrow \vec{A} \perp \vec{B}$

Problem 2.10

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} -2 & -3 & 1 \\ 2 & -5 & 3 \\ 4 & 2 & 6 \end{vmatrix}$$

$$= 48$$

Problem 2.11

$$\vec{C} = C\vec{a}_n = \vec{A} \times \vec{B}$$

$$= 2\vec{a}_x + 6\vec{a}_y + 10\vec{a}_z$$

$$C = \sqrt{2^2 + 6^2 + 10^2} = 11.832$$

$$\vec{a}_n = \frac{\vec{C}}{C} = 0.17\vec{a}_x + 0.51\vec{a}_y + 0.85\vec{a}_z$$

Problem 2.12 $\vec{A} = \vec{PQ} = 2\vec{a}_x + \vec{a}_y + 4\vec{a}_z$, $\vec{B} = \vec{PS} = 4\vec{a}_x + 6\vec{a}_y + 8\vec{a}_z$
 $\vec{A} \times \vec{B} = -16\vec{a}_x + 8\vec{a}_z \Rightarrow \text{Area} = \frac{1}{2} |\vec{A} \times \vec{B}| = 8.94$

Problem 2.13 $\vec{A} = 4\vec{a}_x - 3\vec{a}_y + \vec{a}_z$, $\vec{B} = 2\vec{a}_x + \vec{a}_y - \vec{a}_z$
 $\cos \theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{8-3-1}{AB} = 0.39$ $A = 5.1$, $B = 2.45$
 $\theta = 71.33^\circ$

Problem 2.14 $\vec{A} = 3\vec{a}_\rho + 5\vec{a}_\phi - 4\vec{a}_z$, $\vec{B} = 2\vec{a}_\rho + 4\vec{a}_\phi + 3\vec{a}_z$
 $\vec{A} + \vec{B} = 5\vec{a}_\rho + 9\vec{a}_\phi - \vec{a}_z$ $A = 7.071$ $B = 5.385$
 $\vec{A} \cdot \vec{B} = 6 + 20 - 12 = 14$ $\cos \alpha = \frac{\vec{A} \cdot \vec{B}}{AB} = 0.368 \Rightarrow \alpha = 68.43^\circ$
 $\vec{A} \times \vec{B} = 31\vec{a}_\rho - 17\vec{a}_\phi + 2\vec{a}_z$
 $|\vec{A} \times \vec{B}| = 35.412$ $\vec{a}_n = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = 0.875\vec{a}_\rho - 0.48\vec{a}_\phi + 0.056\vec{a}_z$

Scalar projection of \vec{A} onto \vec{B} : $\vec{A} \cdot \vec{a}_B = \frac{\vec{A} \cdot \vec{B}}{B} = 2.6$

Vector projection of \vec{A} onto \vec{B} : $2.6\vec{a}_B = 0.483[2\vec{a}_\rho + 4\vec{a}_\phi + 3\vec{a}_z]$

Problem 2.15 $P(5, 30^\circ, 5) = P(5 \cos 30^\circ, 5 \sin 30^\circ, 5) = P(4.33, 2.5, 5)$
 $Q(2, 60^\circ, 4) = Q(2 \cos 60^\circ, 2 \sin 60^\circ, 4) = Q(1, 1.732, 4)$
 $\vec{r} = \vec{PQ} = -3.33\vec{a}_x - 0.768\vec{a}_y - \vec{a}_z \Rightarrow r = PQ = 3.56$

Problem 2.16 $\vec{A} = 2\vec{a}_\rho + 3\vec{a}_\phi \in P(1, 90^\circ, 2) \Rightarrow \vec{A} = 2\vec{a}_y - 3\vec{a}_x$
 $\vec{B} = -3\vec{a}_\rho + 10\vec{a}_z \in (2, \pi, 3) \Rightarrow \vec{B} = 3\vec{a}_x + 10\vec{a}_z$
 $\vec{A} + \vec{B} = 2\vec{a}_y + 10\vec{a}_z$ $\vec{A} \cdot \vec{B} = -9$ $A = \sqrt{13}$ $B = \sqrt{109}$
 $\cos \theta = \vec{A} \cdot \vec{B} / AB \Rightarrow \theta = 103.83^\circ$ $\vec{A} \times \vec{B} = 20\vec{a}_x + 30\vec{a}_y - 6\vec{a}_z$

Problem 2.17 $\vec{A} = -7\vec{a}_r + 2\vec{a}_\theta + \vec{a}_\phi$, $\vec{B} = \vec{a}_r - 2\vec{a}_\theta + 4\vec{a}_\phi$
 $2\vec{A} - 3\vec{B} = -17\vec{a}_r + 10\vec{a}_\theta - 10\vec{a}_\phi$ $A = 7.3485$ $B = 4.5826$
 $\vec{A} \cdot \vec{B} = -7 - 4 + 4 = -7$ $\vec{A} \times \vec{B} = 10\vec{a}_r + 29\vec{a}_\theta + 12\vec{a}_\phi$
 $\cos \alpha = \vec{A} \cdot \vec{B} / AB = -0.208 \Rightarrow \alpha = 108^\circ$
 $\vec{a}_n = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = 0.304\vec{a}_r + 0.88\vec{a}_\theta + 0.364\vec{a}_\phi$

Problem 2.18 Transform \vec{A} and \vec{B} . $P(2, 45^\circ, 45^\circ)$, $Q(10, 90^\circ, 90^\circ)$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin 45^\circ \cos 45^\circ & \cos 45^\circ \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ \sin 45^\circ & \cos 45^\circ \sin 45^\circ & \cos 45^\circ \\ \cos 45^\circ & -\sin 45^\circ & 0 \end{bmatrix} \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$$

$$A_x = -3.207 \quad A_y = -1.793 \quad A_z = -6.363$$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \Rightarrow \begin{aligned} B_x &= -4, B_y = 1, B_z = 2 \\ \vec{A} &= -3.207 \vec{a}_x - 1.793 \vec{a}_y - 6.363 \vec{a}_z \\ \vec{B} &= -4 \vec{a}_x + \vec{a}_y + 2 \vec{a}_z \end{aligned}$$

$$2\vec{A} - 3\vec{B} = 5.586 \vec{a}_x - 6.586 \vec{a}_y - 18.726 \vec{a}_z, \quad \vec{A} \cdot \vec{B} = -1.691$$

$$\vec{A} \times \vec{B} = 2.777 \vec{a}_x + 31.866 \vec{a}_y - 10.379 \vec{a}_z, \quad A = 7.348, B = 4.583$$

$$|\vec{A} \times \vec{B}| = 33.629 \Rightarrow \vec{a}_n = \frac{\vec{A} \times \vec{B}}{33.629} = 0.083 \vec{a}_x + 0.948 \vec{a}_y - 0.309 \vec{a}_z$$

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{AB} = -0.05 \Rightarrow \theta = 92.88^\circ$$

Problem 2.19 $P(10, \pi/4, \pi/3) \Rightarrow P(3.54, 6.12, 7.07)$

$$Q(2, \pi/2, \pi) \Rightarrow Q(-2, 0, 0)$$

$$\vec{PQ} = -5.54 \vec{a}_x - 6.12 \vec{a}_y - 7.07 \vec{a}_z \Rightarrow |\vec{PQ}| = 10.87$$

Problem 2.20 $f = 12xy + z$ $P(0, 0, 0) \rightarrow Q(1, 1, 0) \quad z=0 \quad dz=0$

Straight line: $y=x$

$$\begin{aligned} a) \int_C f d\vec{l} &= \int_C 12xy [dx \vec{a}_x + dy \vec{a}_y] \\ &= \vec{a}_x \int_0^1 12xy dx + \vec{a}_y \int_0^1 12xy dy \\ &= \vec{a}_x \int_0^1 12x^2 dx + \vec{a}_y \int_0^1 12y^2 dy \\ &= 4 \vec{a}_x + 4 \vec{a}_y \end{aligned}$$

$$\begin{aligned} b) \int_C f d\vec{l} &= \int_C f \sqrt{dx^2 + dy^2} \\ &= \int_C f \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= 16.97 \int_0^1 xy dx = 16.97 \int_0^1 x^2 dx \\ &= 5.66 \end{aligned}$$

Problem 2.21 $\vec{E} = \frac{10}{r} \vec{a}_r$

$$a = 10 \text{ cm} = 0.1 \text{ m}$$

$$b = 80 \text{ cm} = 0.8 \text{ m}$$

$$\begin{aligned} V_{ab} &= - \int_b^a \vec{E} \cdot d\vec{l} = - \int_{0.8}^{0.1} \frac{10}{r} dr \\ &= -10 \ln \left[\frac{0.1}{0.8} \right] = 10 \ln(8) \\ &= 20.79 \text{ V} \end{aligned}$$

Problem 2.22 $n_e = 300 \rho \cos^2 \phi$ electrons/m³ disc radius = 20 m

$$N = \int_V n_e dV = 300 \int_0^{20} \rho^2 d\rho \int_0^{2\pi} \cos^2 \phi d\phi$$

$$= 100 \rho^3 \Big|_0^{20} \left[\frac{1}{2} \phi + \frac{1}{4} \sin 2\phi \right]_0^{2\pi} = 2,513,274 \text{ electrons}$$

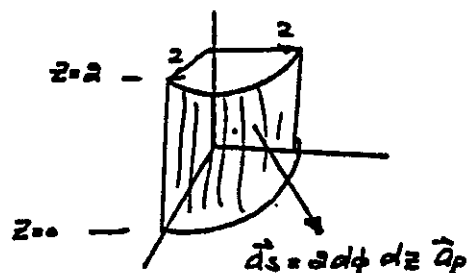
$$Q = -1.6 \times 10^{-19} \times 2,513,274 = -4.02 \times 10^{-13} \text{ C or } 0.402 \text{ pC}$$

Problem 2.23

$$f = xyz \quad x = 2 \cos \phi \quad y = 2 \sin \phi$$

$$\int_V f dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 8 \sin \phi \cos \phi z dz d\phi$$

$$= 8 \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2} \left[\frac{z^2}{2} \right]_0^1 = 2$$



Problem 2.24

$$\vec{F} = x^3 \vec{a}_x + x^2 y \vec{a}_y + x^2 z \vec{a}_z$$

Surface 1: $d\vec{s}_1 = \rho d\phi d\rho \vec{a}_z$

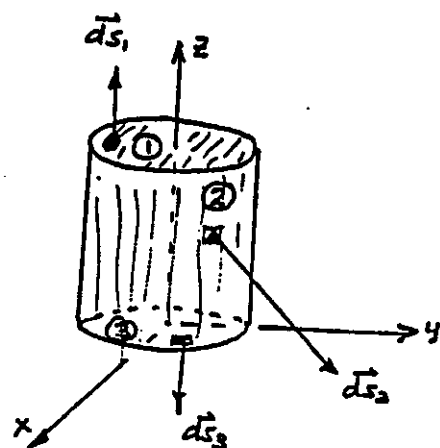
$$\int_{S_1} \vec{F} \cdot d\vec{s}_1 = \iint x^2 z \rho d\rho d\phi$$

$$z = 2 \quad x = \rho \cos \phi$$

Thus

$$\int_{S_1} \vec{F} \cdot d\vec{s}_1 = 2 \int_0^4 \rho^3 d\rho \int_0^{2\pi} \cos^2 \phi d\phi$$

$$= 128\pi$$



Surface 3: $d\vec{s}_3 = -\rho d\rho d\phi \vec{a}_z$

$$z = 0$$

$$\int_{S_3} \vec{F} \cdot d\vec{s}_3 = 0$$

Surface 2: $d\vec{s}_2 = 4 d\phi dz \vec{a}_\rho$

$$\int_{S_2} \vec{F} \cdot d\vec{s}_2 = 256 \int_0^{2\pi} \cos^2 \phi d\phi \int_0^2 dz$$

$$= 512\pi$$

$$\vec{F} \cdot d\vec{s}_2 = (x^3 \vec{a}_x \cdot \vec{a}_\rho + x^2 y \vec{a}_y \cdot \vec{a}_\rho) 4 d\phi dz$$

$$x = 4 \cos \phi \quad y = 4 \sin \phi$$

$$\vec{F} \cdot d\vec{s}_2 = [64 \cos^3 \phi + 64 \cos^2 \phi \sin^2 \phi] \cdot 4 d\phi dz$$

$$= 256 \cos^2 \phi d\phi dz$$

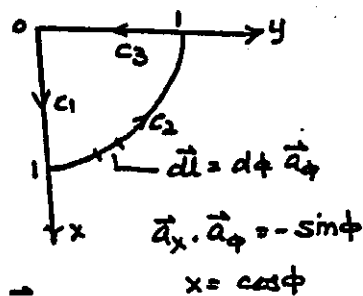
$$\oint \vec{F} \cdot d\vec{s} = 128\pi + 0 + 512\pi = 640\pi$$

Problem 2.25 $\vec{F} = x \vec{a}_x$

Along C_1 : $\int_{C_1} \vec{F} \cdot d\vec{l} = \int_0^1 x dx = 0.5$

Along C_2 : $\int_{C_2} \vec{F} \cdot d\vec{l} = - \int_0^{\pi/2} \sin \phi \cos \phi d\phi = -0.5$

Along C_3 : $\int_{C_3} \vec{F} \cdot d\vec{l} = + \int_1^0 x \vec{a}_x \cdot \vec{a}_y dy = 0 \Rightarrow \oint \vec{F} \cdot d\vec{l} = 0$



Problem 2.26 $\vec{F} = xy \vec{a}_x$ $d\vec{l} = 2 d\theta \vec{a}_\theta$ $\vec{a}_x \cdot \vec{a}_\theta = \cos \theta \cos \phi$

$\vec{F} \cdot d\vec{l} = 2xy \vec{a}_x \cdot \vec{a}_\theta$

$\phi = 60^\circ$

$= 0.5 \cos \theta$

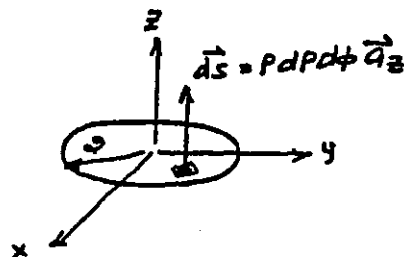
$x = 2 \sin \theta \cos \phi = \sin \theta$

$y = 2 \sin \theta \sin \phi = 1.732 \sin \theta$

$\int_C \vec{F} \cdot d\vec{l} = 1.732 \int_0^\pi \sin^2 \theta \cos \theta d\theta$
 $= 0.577 \sin^3 \theta \Big|_0^\pi = 0$

Problem 2.27 $\vec{D} = (2 + 16\rho^2) \vec{a}_z$

$\int_S \vec{D} \cdot d\vec{s} = \int_0^2 (2 + 16\rho^2) \rho d\rho \int_0^{2\pi} d\phi = 136\pi$



Problem 2.28 $\vec{D} = (2 + 16r^2) \vec{a}_z$ $d\vec{s} = 4 \sin \theta d\theta d\phi \vec{a}_r$

$\int_S \vec{D} \cdot d\vec{s} = \int_0^{\pi/2} (2 + 16 \times 4) 4 \sin \theta \cos \theta d\theta \int_0^{2\pi} d\phi$
 $= 264\pi$

$\vec{a}_z \cdot \vec{a}_r = \cos \theta$

Problem 2.29 $\vec{D} = 10 \cos \phi \vec{a}_\phi$ $d\vec{s} = \rho d\rho dz \vec{a}_z$ $d\vec{s} \cdot \vec{D} = 0$

$\int_S \vec{D} \cdot d\vec{s} = 0$

Problem 2.30

$\vec{D} = 10 \cos \theta \vec{a}_r$ $d\vec{s} = 4 \sin \theta d\theta d\phi \vec{a}_r$

$\int_S \vec{D} \cdot d\vec{s} = 40 \int_0^{\pi/2} \sin \theta \cos \theta d\theta \int_0^{2\pi} d\phi$
 $= 40\pi$

Problem 2.31

$Q = \int_V \rho dv = k \int_0^a r^4 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi$
 $= 0.8\pi k a^5 \quad C$

Problem 2.32

$$\vec{F} = xy^2 \vec{a}_x + (x^2+1)y \vec{a}_y$$

a) $\oint \vec{F} \cdot d\vec{l} = ?$

$$d\vec{l} = 3d\phi \vec{a}_\phi$$

$$\vec{a}_x \cdot \vec{a}_\phi = -\sin\phi$$

$$\vec{F} \cdot d\vec{l} = [3xy^2 d\phi \vec{a}_x \cdot \vec{a}_\phi + (x^2+1)y \vec{a}_y \cdot \vec{a}_\phi](3d\phi)$$

$$\vec{a}_y \cdot \vec{a}_\phi = \cos\phi$$

$$\oint \vec{F} \cdot d\vec{l} = \int_0^{2\pi} (-81 \cos\phi \sin^3\phi + 81 \sin\phi \cos^3\phi + 9 \sin\phi \cos\phi) d\phi$$

$$x = 3 \cos\phi, y = 3 \sin\phi$$

$$= 0$$

b) $\vec{F} \cdot d\vec{s} = 0 \because d\vec{s} = \rho d\phi d\rho \vec{a}_z$ and $F_z = 0$.

Problem 2.33

$$f = x^3 y^2 z \Rightarrow \nabla f = 3x^2 y^2 z \vec{a}_x + 2x^3 y z \vec{a}_y + x^3 y^2 \vec{a}_z$$

$$= 540 \vec{a}_x + 240 \vec{a}_y + 72 \vec{a}_z \quad \mathcal{C}(2,3,5)$$

$$\nabla^2 f = \nabla \cdot (\nabla f) = 6xy^2z + 2x^3z = 620 \quad \mathcal{C}(2,3,5)$$

Problem 2.34

$$\nabla\phi = \frac{1}{\rho} \vec{a}_\phi, \quad \nabla \times [\vec{a}_z \ln \rho] = -\frac{1}{\rho} \vec{a}_\phi$$

a) Hence, $\nabla\phi + \nabla \times (\vec{a}_z \ln \rho) = 0$

$$\nabla(\ln \rho) = \frac{1}{\rho} \vec{a}_\rho \quad \nabla \times (\vec{a}_z \phi) = \frac{1}{\rho} \vec{a}_\rho$$

b) Hence, $\nabla(\ln \rho) - \nabla \times (\vec{a}_z \phi) = 0$

Problem 2.35

$$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2} \vec{a}_r, \quad \nabla\phi = \frac{1}{r \sin\theta} \vec{a}_\phi$$

$$\nabla \times (\cos\theta \nabla\phi) = \nabla \times \left(\frac{\cos\theta}{r \sin\theta} \vec{a}_\phi \right) = -\frac{1}{r^2} \vec{a}_r$$

a) Hence, $\nabla(1/r) - \nabla \times (\cos\theta \nabla\phi) = 0$

$$\nabla\theta = \frac{1}{r} \vec{a}_\theta \Rightarrow \frac{r \nabla\theta}{\sin\theta} = \frac{1}{\sin\theta} \vec{a}_\phi, \quad \nabla \times \left(\frac{r \nabla\theta}{\sin\theta} \right) = \frac{1}{r \sin\theta} \vec{a}_\phi$$

b) Hence $\nabla\phi - \nabla \times \left(\frac{r \nabla\theta}{\sin\theta} \right) = 0$

Problem 2.36

$$\vec{E} = yz \vec{a}_x + xz \vec{a}_y + xy \vec{a}_z$$

$$\nabla \cdot \vec{E} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0 \Rightarrow \text{Solenoidal or Continuous field}$$

$$\nabla \times \vec{E} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \vec{a}_x [x-x] + \vec{a}_y [y-y] + \vec{a}_z [z-z] = 0$$

$\therefore \nabla \times \vec{E} = 0, \vec{E}$ is conservative field

Problem 2.37

$$\text{Let } \vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

$$\nabla \cdot \nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y} = 0$$

$$\nabla \times \nabla f = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \vec{a}_x \left[\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right] + \vec{a}_y \left[\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right] + \vec{a}_z \left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right] = 0$$

Problem 2.38

$$a) \quad \Phi = V_0 \phi \ln\left(\frac{\rho}{a}\right) \quad \vec{E} = -\nabla \Phi = - \left[\frac{\partial \Phi}{\partial \rho} \vec{a}_\rho + \frac{1}{\rho} \frac{\partial \Phi}{\partial \phi} \vec{a}_\phi \right] = - \frac{V_0 \phi}{\rho} \vec{a}_\rho - \frac{V_0}{\rho} \ln\left(\frac{\rho}{a}\right) \vec{a}_\phi$$

$$\nabla \cdot \vec{E} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (-V_0 \phi) + \frac{1}{\rho} \frac{\partial}{\partial \phi} \left(-\frac{V_0}{\rho} \ln\left(\frac{\rho}{a}\right) \right) = 0 \Rightarrow$$

$$\rho_v = \epsilon_0 \nabla \cdot \vec{E} = 0$$

$$b) \quad \Phi = V_0 r \cos \theta \quad \nabla \Phi = \frac{\partial}{\partial r} (V_0 r \cos \theta) \vec{a}_r + \frac{1}{r} \frac{\partial}{\partial \theta} (V_0 r \cos \theta) \vec{a}_\theta$$

$$= V_0 \cos \theta \vec{a}_r - V_0 \sin \theta \vec{a}_\theta$$

$$\vec{E} = -\nabla \Phi = -V_0 \cos \theta \vec{a}_r + V_0 \sin \theta \vec{a}_\theta$$

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (-V_0 r^2 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (V_0 \sin^2 \theta)$$

$$= -\frac{2}{r} V_0 \cos \theta + \frac{2}{r} V_0 \cos \theta = 0$$

$$\rho_v = \epsilon_0 \nabla \cdot \vec{E} = 0$$

$$c) \quad \Phi = V_0 r \sin \theta$$

$$\nabla \Phi = V_0 \sin \theta \vec{a}_r + V_0 \cos \theta \vec{a}_\theta$$

$$\vec{E} = -\nabla \Phi = -V_0 \sin \theta \vec{a}_r - V_0 \cos \theta \vec{a}_\theta$$

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (-V_0 r^2 \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (-V_0 \sin \theta \cos \theta)$$

$$= -\frac{2}{r} V_0 \sin \theta - \frac{V_0}{r \sin \theta} [\cos^2 \theta - \sin^2 \theta] = -\frac{V_0}{r \sin \theta}$$

$$\rho_v = \epsilon_0 \nabla \cdot \vec{E} = -\frac{V_0 \epsilon_0}{r \sin \theta}$$

Problem 2.39

$$\begin{aligned} \text{a) } \nabla(fg) &= \vec{a}_x \left[f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x} \right] + \vec{a}_y \left[f \frac{\partial g}{\partial y} + g \frac{\partial f}{\partial y} \right] + \vec{a}_z \left[f \frac{\partial g}{\partial z} + g \frac{\partial f}{\partial z} \right] \\ &= f \left[\frac{\partial g}{\partial x} \vec{a}_x + \frac{\partial g}{\partial y} \vec{a}_y + \frac{\partial g}{\partial z} \vec{a}_z \right] + g \left[\frac{\partial f}{\partial x} \vec{a}_x + \frac{\partial f}{\partial y} \vec{a}_y + \frac{\partial f}{\partial z} \vec{a}_z \right] \\ &= f \nabla g + g \nabla f \end{aligned}$$

$$\text{b) } f\vec{A} = fA_x \vec{a}_x + fA_y \vec{a}_y + fA_z \vec{a}_z$$

$$\begin{aligned} \nabla \cdot (f\vec{A}) &= \frac{\partial}{\partial x} (fA_x) + \frac{\partial}{\partial y} (fA_y) + \frac{\partial}{\partial z} (fA_z) \\ &= A_x \frac{\partial f}{\partial x} + f \frac{\partial A_x}{\partial x} + A_y \frac{\partial f}{\partial y} + f \frac{\partial A_y}{\partial y} + A_z \frac{\partial f}{\partial z} + f \frac{\partial A_z}{\partial z} \\ &= (A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z) \cdot \left(\frac{\partial f}{\partial x} \vec{a}_x + \frac{\partial f}{\partial y} \vec{a}_y + \frac{\partial f}{\partial z} \vec{a}_z \right) + f \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \\ &= \vec{A} \cdot \nabla f + f \nabla \cdot \vec{A} \end{aligned}$$

$$\text{c) } \nabla \times f\vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fA_x & fA_y & fA_z \end{vmatrix}$$

$$\begin{aligned} &= \vec{a}_x \left[\frac{\partial}{\partial y} (fA_z) - \frac{\partial}{\partial z} (fA_y) \right] + \vec{a}_y \left[\frac{\partial}{\partial z} (fA_x) - \frac{\partial}{\partial x} (fA_z) \right] \\ &\quad + \vec{a}_z \left[\frac{\partial}{\partial x} (fA_y) - \frac{\partial}{\partial y} (fA_x) \right] \end{aligned}$$

$$= \vec{a}_x \left[f \frac{\partial A_z}{\partial y} + A_z \frac{\partial f}{\partial y} - f \frac{\partial A_y}{\partial z} - A_y \frac{\partial f}{\partial z} \right]$$

$$+ \vec{a}_y \left[f \frac{\partial A_x}{\partial z} + A_x \frac{\partial f}{\partial z} - f \frac{\partial A_z}{\partial x} - A_z \frac{\partial f}{\partial x} \right]$$

$$+ \vec{a}_z \left[f \frac{\partial A_y}{\partial x} + A_y \frac{\partial f}{\partial x} - f \frac{\partial A_x}{\partial y} - A_x \frac{\partial f}{\partial y} \right]$$

$$= f \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \vec{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \vec{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \vec{a}_z \right]$$

$$+ \left(\frac{\partial f}{\partial y} A_z - \frac{\partial f}{\partial z} A_y \right) \vec{a}_x + \left(\frac{\partial f}{\partial z} A_x - \frac{\partial f}{\partial x} A_z \right) \vec{a}_y$$

$$+ \left(\frac{\partial f}{\partial x} A_y - \frac{\partial f}{\partial y} A_x \right) \vec{a}_z$$

$$= f \nabla \times \vec{A} + \nabla f \times \vec{A}$$

Problem 2.40 $x = \rho \cos \phi \Rightarrow dx = \cos \phi d\rho - \rho \sin \phi d\phi$ ①

$y = \rho \sin \phi \Rightarrow dy = \sin \phi d\rho + \rho \cos \phi d\phi$ ②

From ① and ②: $d\rho = \cos \phi dx + \sin \phi dy$, and $d\phi = -\frac{1}{\rho} \sin \phi dx + \frac{1}{\rho} \cos \phi dy$

However, $d\rho = \frac{\partial \rho}{\partial x} dx + \frac{\partial \rho}{\partial y} dy \Rightarrow \frac{\partial \rho}{\partial x} = \cos \phi$ and $\frac{\partial \rho}{\partial y} = \sin \phi$

$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \Rightarrow \frac{\partial \phi}{\partial x} = -\frac{1}{\rho} \sin \phi$ and $\frac{\partial \phi}{\partial y} = \frac{1}{\rho} \cos \phi$

Thus, $\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x} \frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} = \cos \phi \frac{\partial}{\partial \rho} - \frac{1}{\rho} \sin \phi \frac{\partial}{\partial \phi}$

$\frac{\partial}{\partial y} = \frac{\partial \rho}{\partial y} \frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi} = \sin \phi \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cos \phi \frac{\partial}{\partial \phi}$

Problem 2.41

$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \right) = \left(\cos \phi \frac{\partial}{\partial \rho} - \frac{1}{\rho} \sin \phi \frac{\partial}{\partial \phi} \right) \left(\cos \phi \frac{\partial}{\partial \rho} - \frac{1}{\rho} \sin \phi \frac{\partial}{\partial \phi} \right)$

$= \cos^2 \phi \frac{\partial^2}{\partial \rho^2} + \frac{\sin^2 \phi}{\rho} \frac{\partial}{\partial \rho} - \frac{\sin \phi \cos \phi}{\rho} \frac{\partial^2}{\partial \rho \partial \phi} + \frac{\sin \phi \cos \phi}{\rho^2} \frac{\partial}{\partial \phi}$
 $- \frac{\sin \phi \cos \phi}{\rho} \frac{\partial^2}{\partial \rho \partial \phi} + \frac{\sin \phi}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\sin \phi \cos \phi}{\rho^2} \frac{\partial}{\partial \phi}$

$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} \right) = \left(\sin \phi \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cos \phi \frac{\partial}{\partial \phi} \right) \left(\sin \phi \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cos \phi \frac{\partial}{\partial \phi} \right)$

$= \sin^2 \phi \frac{\partial^2}{\partial \rho^2} + \frac{\sin \phi \cos \phi}{\rho} \frac{\partial^2}{\partial \rho \partial \phi} - \frac{\sin \phi \cos \phi}{\rho^2} \frac{\partial}{\partial \phi}$
 $+ \frac{\sin \phi \cos \phi}{\rho} \frac{\partial^2}{\partial \phi \partial \rho} + \frac{\cos^2 \phi}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \cos^2 \phi \frac{\partial^2}{\partial \phi^2} - \frac{\sin \phi \cos \phi}{\rho^2} \frac{\partial}{\partial \phi}$

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} = \frac{1}{\rho^2} \left[\rho \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial \phi^2} \right]$

Problem 2.42 $\vec{E} = E_0 \cos \theta \vec{a}_r - E_0 \sin \theta \vec{a}_\theta$

$\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_0 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (-E_0 \sin^2 \theta) = 0$

$\nabla \times \vec{E} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \vec{a}_r & r \vec{a}_\theta & r \sin \theta \vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ E_0 \cos \theta & -E_0 r \sin \theta & 0 \end{vmatrix}$

$= \frac{\vec{a}_\phi}{r^2 \sin \theta} \cdot r \sin \theta [-E_0 \sin \theta + E_0 \sin \theta] = 0$

Problem 2.43

$\nabla \cdot \vec{E} = 0$ from Problem 2.42. Thus, $\int_V \nabla \cdot \vec{E} \, dv = 0$

$$\oint_S \vec{E} \cdot d\vec{s} = \int_0^\pi (E_0 \cos \theta) r^2 \sin \theta \, d\theta \int_0^{2\pi} d\phi \quad \text{when } r=b$$

$$= b^2 E_0 \pi \sin^2 \theta \Big|_0^\pi = 0$$

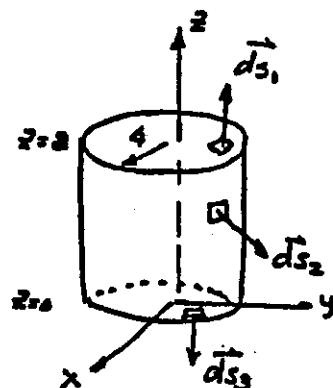
Problem 2.44

$$\vec{F} = x^3 \vec{a}_x + x^2 y \vec{a}_y + x^2 z \vec{a}_z$$

$$\nabla \cdot \vec{F} = 3x^2 + x^2 + x^2 = 5x^2$$

$$\int_V \nabla \cdot \vec{F} \, dv = \int_V 5x^2 \, dv = 5 \int_0^4 \rho^3 \, d\rho \int_0^{2\pi} \cos^2 \phi \, d\phi \int_0^2 dz$$

$$= \frac{5}{4} \rho^4 \Big|_0^4 (\pi)(2) = 640\pi$$



Surface S_1 : $\int_{S_1} \vec{F} \cdot d\vec{s}_1 = \int_{S_1} x^2 z \, \rho \, d\rho \, d\phi$ where $z=2$

$$= 2 \int_0^4 \rho^3 \, d\rho \int_0^{2\pi} \cos^2 \phi \, d\phi = 128\pi$$

Surface S_3 : $\int_{S_3} \vec{F} \cdot d\vec{s}_3 = \int_{S_3} x^2 z \, \rho \, d\rho \, d\phi = 0 \quad \because z=0$

Surface S_2 : $\int_{S_2} \vec{F} \cdot d\vec{s}_2 = \int_{S_2} [x^3 \vec{a}_x \cdot \vec{a}_\rho + x^2 y \vec{a}_y \cdot \vec{a}_\rho] \, \rho \, d\phi \, dz \quad \rho=4$

$$= 256 \int_0^{2\pi} \cos^4 \phi \, d\phi \int_0^2 dz + 256 \int_0^{2\pi} \cos^2 \phi \sin^2 \phi \, d\phi \int_0^2 dz$$

$$= 256 \int_0^{2\pi} \cos^2 \phi \, d\phi \int_0^2 dz = 256 (\pi)(2) = 512\pi$$

$$\oint \vec{F} \cdot d\vec{s} = 128\pi + 512\pi + 0 = 640\pi$$

Problem 2.45

$$\vec{A} = (12 + 6\rho^2) z \vec{a}_z$$

$$\nabla \cdot \vec{A} = 12 + 6\rho^2$$

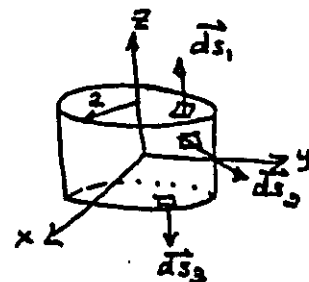
$$\int_V \nabla \cdot \vec{A} \, dv = \int_0^2 (12 + 6\rho^2) \rho \, d\rho \int_0^{2\pi} d\phi \int_{-1}^1 dz = 192\pi$$

$$\int_{S_1} \vec{A} \cdot d\vec{s}_1 = \int_0^2 (12 + 6\rho^2) \rho \, d\rho \int_0^{2\pi} d\phi = 96\pi$$

$$\int_{S_2} \vec{A} \cdot d\vec{s}_2 = 0$$

$$\int_{S_3} \vec{A} \cdot d\vec{s}_3 = \int_0^2 (12 + 6\rho^2) \rho \, d\rho \int_0^{2\pi} d\phi = 96\pi$$

$$\oint \vec{A} \cdot d\vec{s} = 192\pi = \int_V \nabla \cdot \vec{A} \, dv$$



$$d\vec{s}_1 = \rho \, d\rho \, d\phi \vec{a}_z \quad \text{at } z=1$$

$$d\vec{s}_2 = 2 \, d\phi \, dz \vec{a}_\rho \quad \text{at } \rho=2$$

$$d\vec{s}_3 = -\rho \, d\rho \, d\phi \vec{a}_z \quad \text{at } z=-1$$

Problem 2.46 $\vec{F} = 3y^2 \vec{a}_x + 4z \vec{a}_y + 6y \vec{a}_z$

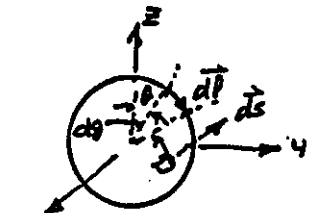
$$\left. \begin{aligned} \vec{a}_z \cdot \vec{a}_\theta &= \cos(90^\circ + \theta) = -\sin\theta & z &= 2\cos\theta \\ \vec{a}_y \cdot \vec{a}_\theta &= \cos\theta & y &= 2\sin\theta \\ \vec{a}_x \cdot \vec{a}_\theta &= 0 \end{aligned} \right\} \text{ at } r=2$$

$$\oint_C \vec{F} \cdot d\vec{l} = \oint_C (6y^2 d\theta \vec{a}_x \cdot \vec{a}_\theta + 8z d\theta \vec{a}_y \cdot \vec{a}_\theta + 12y d\theta \vec{a}_z \cdot \vec{a}_\theta)$$

$$= \int_0^{2\pi} 16 \cos^2\theta d\theta - \int_0^{2\pi} 24 \sin^2\theta d\theta = -8\pi$$

$$(\nabla \times \vec{F}) \cdot \vec{a}_x = +2 \Rightarrow \int_S \nabla \times \vec{F} \cdot d\vec{s} = -2 \int_0^2 r dr \int_0^{2\pi} d\theta = -8\pi$$

$$\text{Thus, } \oint_C \vec{F} \cdot d\vec{l} = -8\pi = \int_S (\nabla \times \vec{F}) \cdot d\vec{s}$$



$$d\vec{l} = 2 d\theta \vec{a}_\theta$$

$$d\vec{s} = -r dr d\theta \vec{a}_x$$

Problem 2.47 $\vec{F} = \frac{x}{\rho} \vec{a}_x$

$$d\vec{s} = \rho d\rho d\phi \vec{a}_z$$

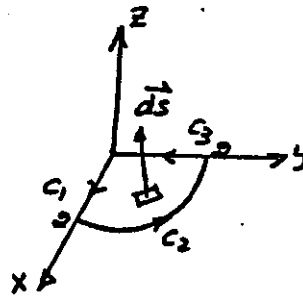
$$\nabla \times \vec{F} \cdot d\vec{s} = \left[\vec{a}_y \frac{\partial}{\partial z} \left(\frac{x}{\rho} \right) - \vec{a}_z \frac{\partial}{\partial y} \left(\frac{x}{\rho} \right) \right] \cdot \rho d\rho d\phi \vec{a}_z$$

$$= - \frac{\partial}{\partial y} \left(\frac{x}{\rho} \right) \rho d\rho d\phi$$

$$x = \rho \cos\phi, y = \rho \sin\phi, \rho = \sqrt{x^2 + y^2}$$

$$\frac{\partial}{\partial y} \left(\frac{x}{\rho} \right) = \frac{\partial}{\partial y} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) = - \frac{xy}{\rho^3} = - \frac{\sin\phi \cos\phi}{\rho}$$

$$\int_S \nabla \times \vec{F} \cdot d\vec{s} = \int_0^2 \rho d\rho \int_0^{\pi/2} \sin\phi \cos\phi d\phi = 1$$



$$\text{Path } C_1: d\vec{l}_1 = dx \vec{a}_x \Rightarrow \int_{C_1} \vec{F} \cdot d\vec{l}_1 = \int_0^2 \frac{x}{\rho} dx = \int_0^2 dx = 2 \quad (\rho=x \because \phi=0)$$

$$\text{Path } C_2: d\vec{l}_2 = 2 d\phi \vec{a}_\phi \quad \rho=2 \quad \vec{a}_x \cdot \vec{a}_\phi = -\sin\phi$$

$$\vec{F} \cdot d\vec{l}_2 = \frac{x}{2} \cdot 2 d\phi \vec{a}_x \cdot \vec{a}_\phi$$

$$x = 2 \cos\phi$$

$$\int_{C_2} \vec{F} \cdot d\vec{l}_2 = -2 \int_0^{\pi/2} \sin\phi \cos\phi d\phi = -1$$

$$\text{Path } C_3: d\vec{l}_3 = dy \vec{a}_y \quad \vec{F} \cdot d\vec{l}_3 = 0 \quad \int_{C_3} \vec{F} \cdot d\vec{l}_3 = 0$$

$$\text{Thus, } \oint \vec{F} \cdot d\vec{l} = 2 - 1 + 0 = 1 = \int_S \nabla \times \vec{F} \cdot d\vec{s}$$

Problem 2.48

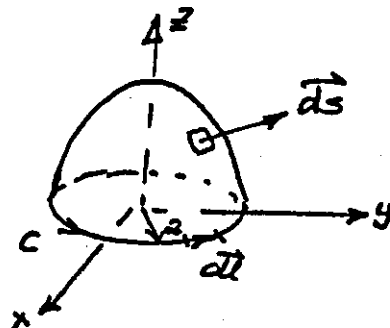
$$\vec{F} = 100 \cos \theta \vec{a}_r$$

on path C , $d\vec{l} = 2 d\phi \vec{a}_\phi$

$$\oint_C \vec{F} \cdot d\vec{l} = 0$$

$$\nabla \times \vec{F} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \vec{a}_r & r \vec{a}_\theta & r \sin \theta \vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 100 \cos \theta & 0 & 0 \end{vmatrix}$$

$$= \frac{100}{r} \sin \theta \vec{a}_\phi \Rightarrow \int_S \nabla \times \vec{F} \cdot d\vec{s} = 0$$



$$d\vec{s} = r^2 \sin \theta d\theta d\phi \vec{a}_r \text{ at } r=2$$

Problem 2.49 $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $-\frac{1}{2} \leq y \leq \frac{1}{2}$, $-\frac{1}{2} \leq z \leq \frac{1}{2}$

$$f = x^2 \Rightarrow \nabla f = 2x \vec{a}_x \text{ and } \nabla^2 f = 2$$

$$g = y^2 \Rightarrow \nabla g = 2y \vec{a}_y \text{ and } \nabla^2 g = 2 \quad \nabla f \cdot \nabla g = 0 \Rightarrow$$

$$\int_V f \nabla^2 g \, dv = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \int_{-\frac{1}{2}}^{\frac{1}{2}} dz = \frac{1}{6} \quad \int_V \nabla f \cdot \nabla g \, dv = 0$$

$$\oint_S f \nabla g \cdot d\vec{s} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} 2x^2 y \, dx \int_{-\frac{1}{2}}^{\frac{1}{2}} dz \Big|_{y=\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} 2x^2 y \, dx \int_{-\frac{1}{2}}^{\frac{1}{2}} dz \Big|_{y=-\frac{1}{2}} = \frac{1}{6}$$

Hence, $\int_V f \nabla^2 g \, dv + \int_V \nabla f \cdot \nabla g \, dv = \oint_S f \nabla g \cdot d\vec{s}$. Green's 1st Identity =

$$\int_V g \nabla^2 f \, dv = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} 2y^2 \, dy \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{-\frac{1}{2}}^{\frac{1}{2}} dz = \frac{1}{6} \Rightarrow \int_V (f \nabla^2 g - g \nabla^2 f) \, dv = 0 \quad (1)$$

$$\oint_S g \nabla f \cdot d\vec{s} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} 2xy^2 \, dy \int_{-\frac{1}{2}}^{\frac{1}{2}} dz \Big|_{x=\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} 2xy^2 \, dy \int_{-\frac{1}{2}}^{\frac{1}{2}} dz \Big|_{x=-\frac{1}{2}} = \frac{1}{6}$$

$$\oint [f \nabla g - g \nabla f] \cdot d\vec{s} = 0 \quad (2)$$

Thus, from (1) and (2),

$$\int_V (f \nabla^2 g - g \nabla^2 f) \, dv = \oint_S (f \nabla g - g \nabla f) \cdot d\vec{s}, \text{ Green's 2nd Identity}$$

